TEMPERATURE FIELD CALCULATION FROM BODY DEFORMATION MEASUREMENTS

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A method is considered for solution of the converse thermoelasticity problem of finding the temperature field on the basis of measurements of deformations at certain points of the body.

The problems of thermal conductivity theory are usually related to finding a temperature field on the basis of specified boundary heat-exchange conditions. In practice, situations occur where determination of these conditions is impossible or very difficult, and it is necessary to use certain indirect measurements, in particular, measurements of body deformation, to determine the temperature field. In this case the problem of finding the temperature field is in essence a converse thermoelasticity problem. In a number of cases the solution of such a problem is the basis for making automation of experimental measurements mathematically possible.

In the present study we will consider a numerical method for reestablishing the temperature field from measurements of the deformation of a plate ($0 \leqslant z \leqslant L$) of arbitrary form. The plate is free, i.e., there are no stresses on the boundaries $z = 0$ and $z = L$. The plate deformation is caused by a change in temperature T over thickness z and time τ, while the effect of deformation on temperature and inertial effects may be neglected. The physical characteristics of the plate material are specified. The temperature at the initial time $\tau = 0$ is known

$$T(z,\ 0) = \varphi(z), \tag{1}$$

as is the temperature on the lower boundary

$$T(0,\ \tau) = \psi(\tau). \tag{2}$$

Moreover, one of the components of the displacement, say that in the z direction, is known for an arbitrary point of the body with coordinates $x = x_0$, $y = y_0$, $z = z_0$ relative to a fixed point

$$w(x_0,\ y_0,\ z_0,\ \tau) = \psi_1(\tau). \tag{3}$$

It is necessary to determine the temperature $T(L, \tau)$ on the upper boundary ($z = L$) of the plate

$$T(L,\ \tau) = T_b(\tau). \tag{4}$$

If the function $T_b(\tau)$ is found in some approximation, then determination of the temperature fields, stresses, or deformations given the conditions of Eqs. (1), (2), (4) becomes a direct thermoelasticity problem, the solution of which presents no difficulties.

Below we will consider a method of finding the functions $T_b(\tau)$ based on the condition that the calculated displacement component w_c corresponding to the function at the point $(x_0,\ y_0,\ z_0)$ must coincide with the specified functions $\psi_1(\tau)$ at a set of points Q, i.e.,

$$|w_c(x_0,\ y_0,\ z_0,\ \tau) - \psi_1(\tau)| < \varepsilon, \quad \tau \in Q, \tag{5}$$

where ϵ is some small number, which is taken to be $10^{-9} \cdot \max|\psi_1(\tau)|$ in the calculations. The set Q can be defined as the minimum number of points in the range of variation of the variable τ, which for all measured displacement values ensures fulfillment of the condition

$$\|w_c(x_0,\ y_0,\ z_0,\ \tau) - \psi_1(\tau)\| \leqslant \delta, \tag{6}$$

where δ is the measurement uncertainty of the original data, in particular $\psi_1(\tau)$.

The search for the function $T_b(\tau)$ involves multiple calculation of temperature fields and displacements using conditions (1), (2), (4). In the method considered here it is not significant whether this direct thermoelasticity problem is solved numerically or analytically. For practical realization of the method the temperature $T_i^{k+1} \approx T(z_i,\ \tau_{k+1})$ at internal points of the region $z_i = ih,\ i = 1, 2,\ \ldots,\ I,\ h = L/I;\ \tau_k = kl,\ k = 1,\ 2,\ \ldots$, is found with an explicit difference method

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$$T_i^{k+1} = T_i^k + \frac{l}{2h^2 c_i^k \rho_i^k} [(\lambda_i^k + \lambda_{i+1}^k)(T_{i+1}^k - T_i^k) - (\lambda_i^k + \lambda_{i-1}^k)(T_i^k - T_{i-1}^k)],$$
$$i = 1,\ 2,\ \ldots,\ I - 1; \qquad (7)$$

$$T_i^0 = \varphi(z_i),\ T_0^{k+1} = \psi(\tau_{k+1}),\ T_I^{k+1} = T_b(\tau_{k+1}). \qquad (8)$$

The thermal conductivity λ, specific heat c, and density ρ may be functions of the coordinate z, time, or temperature.

The displacements, stresses, and deformations may also be determined on the basis of an explicit difference method of solving the thermoelasticity equations in displacements, using the establishment method [1]. For simplicity, we will assume that the modulus of elasticity E, the thermal expansion coefficient α, and Poisson ratio ν are constants. In this case the relationship between displacements, stresses, and deformations, on the one hand, and the temperature $T(z, \tau)$, on the other, can be expressed analytically [2], with the displacement component along the z axis being

$$w(x,\ y,\ z,\ \tau) = -\frac{6M_T}{EL^3}(x^2 + y^2) + \frac{1}{(1-\nu)E}\Big\{(1 + \nu)\alpha E \int_{\frac{L}{2}}^{z} T dz - \nu\left(\frac{2z}{L} - 1\right) N_T - \frac{3\nu(2z - L)^2}{L^3} M_T\Big\}, \qquad (9)$$

where

$$N_T = \alpha E \int_0^L T dz; \quad M_T = \alpha E \int_0^L Tz dz.$$

The search for $T_b(\tau)$ is performed by steps $\Delta\tau$ along the τ axis, with $\Delta\tau$ being chosen from the condition that the thermal perturbations developing within the period $\Delta\tau$ at the boundary z = L must manifest themselves sufficiently at the boundary z = 0. Control calculations show that the step $\Delta\tau$ must satisfy the inequality $\Delta \mathrm{Fo} = \Delta\tau\lambda/c\rho L^2 \geqslant 0.3$. This limitation does not permit recourse to a linear approximation of the function T_b in the interval $\Delta\tau$. We write the function T_b in the interval $\tau_0^s < \tau < \tau_0^s + \Delta\tau$ in the form of a power series

$$T_b(\tau) = a_0^s + a_1^s(\tau - \tau_0^s) + a_2^s(\tau - \tau_0^s)(\tau - \tau_1^s) + \ldots + a_m^s(\tau - \tau_0^s)(\tau - \tau_1^s)\ldots(\tau - \tau_{m-1}^s) + \ldots \qquad (10)$$

This expansion is convenient in that the terms of the series with ordinal number $r > m$ have no effect on the value of the function T_b at the points $\tau_0^s,\ \tau_1^s,\ \ldots,\ \tau_m^s$. Each coefficient a_m^s of series (10) corresponds to a definite point τ_m^s of the interval $[\tau_0^s,\ \tau_0^s + \Delta\tau]$.

Results of numerical experiments have shown that if the change in the temperature field is not too abrupt, it is sufficient to retain only the first three terms in the series of Eq. (10). For the sake of simplicity, in the future we will consider just this case. The coefficients a_0^s, a_1^s, and a_2^s correspond to the points τ_0^s, $\tau_0^s + \Delta\tau$, and $\tau_0^s + \Delta\tau/2$, respectively.

Initial condition (1) defines the value of $T_b(0) = a_0^0 = \varphi(L)$. We assume that the function $T_b(\tau)$ in the interval $[\tau_0^{s-1},\ \tau_0^{s-1} + \Delta\tau]$ has been found, and that the function must be defined in the interval $[\tau_0^s,\ \tau_0^s + \Delta\tau]$. Numerical experiments have shown that for accuracy and economy of the solution it is desirable to set $\tau_0^s = \tau_0^{s-1} + \Delta\tau/2$. Then $T_b(\tau_0^s) = a_0^s = T_b(\tau_0^{s-1} + \Delta\tau/2)$, i.e., the coefficient a_0^s can be considered a known quantity in finding the function $T_b(\tau)$ in the interval $[\tau_0^s,\ \tau_0^s + \Delta\tau]$. The coefficients a_1^s and a_2^s are determined by iteration, with $a_{1(1)}^s = a_1^{s-1}$ and $a_{2(1)}^s = a_2^{s-1}$ in the first approximation. Assuming that in the interval $[\tau_0^s,\ \tau_0^s + \Delta\tau]$ the function $T_{b(1)}(\tau)$ is known in the first approximation, we solve the direct thermoelasticity problem in the same interval, and use Eq. (9) to define the displacement $w_{(1)}(x_0,\ y_0,\ z_0,\ \tau_0^s + \Delta\tau)$ in a first approximation. The difference between the displacement $w_{(n)}(x_0,\ y_0,\ z_0,\ \tau_m^s)$, $m = 1,\ 2$, calculated in the n-th approximation at the point τ_m^s and the specified displacement $\psi_1(\tau_m^s)$ is used as an unbalance signal to obtain the following approximation of the coefficient $a_{m(n+1)}^s$, which is then set in correspondence to the point τ_m^s:

$$a_{m(n+1)}^s = a_{m(n)}^s + [w_{(n)}(x_0,\ y_0,\ z_0,\ \tau_m^s) - \psi_1(\tau_m^s)]\frac{1}{v_{m(n)}^s}. \qquad (11)$$

Here $v_{m(n)}^s$ is the absolute rate of change of the function $w(x_0,\ y_0,\ z_0,\ \tau_m^s)$ with respect to the quantity a_m^s:

$$v_{m(n)}^s = \left|\frac{w_{(n)}(x_0,\ y_0,\ z_0,\ \tau_m^s) - w_{(n-1)}(x_0,\ y_0,\ z_0,\ \tau_m^s)}{a_{m(n)}^s - a_{m(n-1)}^s}\right|, \quad n = 1,\ 2,\ \ldots.$$

TABLE 1. Comparison of Solution of the Converse Thermoelasticity Problem (CTP) with Perturbed Initial Data to Exact Solution

τ	$T(L, \tau)$	Values of T(L, τ) obtained by solution of CTP			
		$b_8=b_{10}=0$; $\xi=0$	$b_8=0$; $\xi=0$; $b_{10}=0,1$; $b_{11}=10$	$b_8=b_{10}=0$; $\xi=0,1$	$b_8=0,1$; $b_9=100$; $b_{10}=0,1$; $b_{11}=10$; $\xi=0,1$
0,15	1,414	1,417	1,471	1,456	1,511
0,45	1,840	1,879	1,907	1,796	1,823
0,75	1,606	1,642	1,598	1,701	1,657
1,05	0,994	1,010	0,971	1,009	0,969
1,35	0,497	0,484	0,518	0,443	0,777
1,65	0,439	0,426	0,457	0,448	0,500
1,95	0,780	0,765	0,738	0,740	0,723
2,25	1,212	1,214	1,159	1,003	0,148
2,4	1,357	1,369	1,407	1,338	1,376
2,7	1,394	1,416	1,380	1,454	1,418
3	1,152	1,167	1,119	1,150	1,103
3,3	0,847	0,848	0,871	0,833	0,857
3,7	0,704	0,691	0,747	0,619	0,778
3,9	0,792	0,780	0,772	0,753	0,751
4,2	1,008	1,022	0,946	0,987	0,946
4,5	1,179	1,188	1,173	1,203	1,154

This last formula cannot be used to determine the rate $v^s_{m(1)}$, which is taken equal to the corresponding rate v^{s-1}_m in the previous step.

After successive approximations for a^s_1 have achieved satisfaction of Eq. (5) at $\tau = \tau^s_0 + \Delta\tau$, a second approximation is performed for the coefficient a^s_2. The subsequent approximations for a^s_2, as in the case of a^s_1, are performed with use of Eq. (11), based on satisfaction of Eq. (5) at the point $\tau^s_0 + \Delta\tau/2$ of the interval $[\tau^s_0, \tau^s_0 + \Delta\tau]$. After each change of the coefficient a^s_2 an iteration cycle is performed to determine the value a^s_1 ensuring satisfaction of Eq. (5) at the point $\tau^s_0 + \Delta\tau$. Then a subsequent approximation for the coefficient a^s_2 is performed. The search for the function $T_b(\tau)$ in the interval $[\tau^s_0, \tau^s_0 + \Delta\tau]$ is terminated when condition (5) is satisfied at both the point $\tau^s_0 + \Delta\tau$, and the point $\tau^s_0 + \Delta\tau/2$.

To test the method described above, the following numerical experiment was performed. Initially, using Eqs. (7)-(9) the direct thermoelasticity problem was solved for the following initial conditions:

$$\varphi(z) = 1; \quad \psi(\tau) = 1 + b_1\tau; \quad T_b(\tau) = \exp(-b_2\tau)\sin b_3\tau;$$
$$z_0 = \frac{L}{2}; \quad \frac{6\alpha}{L^3}(x_0^2 + y_0^2) = b_4; \quad h = \frac{L}{10} = 0,1; \quad l = 0.3\cdot 10^{-2};$$
$$c\rho = 1; \quad \lambda = b_5 + b_6\tau + b_7\tau^2, \quad b_\gamma = \text{const}, \quad \gamma = 1, 2, \ldots .$$

To establish the effect of inaccuracy in initial data measurement on the inaccuracy of the desired quantities in the functions $\varphi(z)$ and $\psi(\tau)$ and also on the function $\psi_1(\tau) = w(x_0, y_0, z_0, \tau)$ found by solution of the direct problem, perturbations were imposed. The temperature field at the initial moment was perturbed by replacing the function $\varphi(z)$ in Eq. (1) by another function $\varphi(z)(1 + b_8 \sin b_9 z)$, $b_8 = 0-0.15$, $b_9 = 0-100$. The temperature at the boundary z = 0 was perturbed by adding to $\psi(\tau)$ the function $\psi b_{10} \sin b_{11}\tau$. The displacement function $\psi_1(\tau)$ was perturbed by using a pseudorandom number generator, producing numbers distributed by a normal law with mean relative error ξ = 0-0.3.

Then, using the method described, the converse thermoelasticity problem was solved for the perturbed initial data and the function T(L, τ) determined. The latter was compared to the exact function $T_b(\tau)$, which was used for solution of the direct problem.

The numerical experiment for specified values of b_γ on a grid I = 10 with change in Fourier number $\text{Fo} = \lambda\tau/c\rho L^2$ over the interval 0 < Fo < 5 required 10 min of machine time on a BÉSM-4 computer. Each time step $(\tau^s_0, \tau^s_0 + \Delta\tau)$, where $\Delta\text{Fo} = \lambda\Delta\tau/c\rho L^2 = 0.3$, required 4-5 iterations for determination of the coefficients a^s_1 and a^s_2. The error in solving the converse problem for unperturbed initial data was practically the same as the error in solving the direct problem on the same grid. Table I presents the results of calculations for cases where the boundary problems of the converse problem are specified exactly or with a certain uncertainty. Uncertainty in specification of the initial data $\varphi(z)$ proves to have an effect on the solution only in the case of relatively low values of the Fourier number Fo < 0.5.

Numerous calculations have shown that the error of the solution usually does not exceed the error in specification of the boundary conditions over a wide range of variation of the latter. This testifies to the effectiveness of the solution method.

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OPTIMIZATION OF MULTILAYER THERMAL INSULATION

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An iteration method is developed for determination of the thicknesses of layers of a multilayer thermal insulation with minimum mass, with consideration of temperature limitations. The penalty function method is employed.

Coating of surfaces by layers of thermal insulation is a widespread method of protecting thermally stressed construction details from the direct action of a high-temperature medium. One must then select the most rational variant of insulation, i.e., optimize the insulation. Often the mass of the insulating material can be considered as the optimization criterion.

We will consider the problem of heating of a multilayer thermal insulation, consisting of n layers of various materials of thickness h_i, i = 1, 2, ..., n. Thermal contact between layers will be assumed ideal:

$$C^i(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial y}\left(\lambda^i(T)\frac{\partial T}{\partial y}\right),$$

$$Y_{i-1}<y<Y_i,\quad 0<t\leqslant t_c,\quad i=1,\ 2,\ \ldots,\ n, \tag{1}$$

$$T(y,\ 0)=\varphi(y),\quad Y_0\leqslant y\leqslant Y_n, \tag{2}$$

$$-\lambda^1(T)\frac{\partial T(Y_0,\ t)}{\partial y}=q_0(t),\quad t>0, \tag{3}$$

$$-\lambda^n(T)\frac{\partial T(Y_n,\ t)}{\partial y}=q_n(t),\quad t>0, \tag{4}$$

$$T(Y_i-0,\ t)=T(Y_i+0,\ t),\quad i=1,\ 2,\ \ldots,\ n-1,\ t>0, \tag{5}$$

$$\lambda^i(T)\frac{\partial T(Y_i-0,\ t)}{\partial y}=\lambda^{i+1}(T)\frac{\partial T(Y_i+0,\ t)}{\partial y},$$

$$i=1,\ 2,\ \ldots,\ n-1,\quad t>0, \tag{6}$$

where $C^i(T)$, $\lambda^i(T)$, $\varphi(y)$, $q_0(t)$, $q_n(t)$ are known functions.

It is necessary to determine the layer thicknesses $h_i=Y_i-Y_{i-1}$, $i=1,\ 2,\ \ldots,\ n$, which minimize the mass of the thermal insulation with consideration of temperature limitations in the seams between the layers. Thus, it is necessary to find the minimum of the function

$$M(\bar{h})=\sum_{i=1}^{n}\rho_i h_i \tag{7}$$

given Eqs. (1)-(6) and the limitations

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